Introduction to Holomorphic Dynamics

Winter 2021
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$$f_c(z) := z^2 + c$$

QUADRATIC FAMILY
LOGISTIC

$c \in \cancel{\mathbb{C}}\ \mathbb{R}$

$$f_c : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$g_a(x) = a x (1 - x)$$

a = parameter

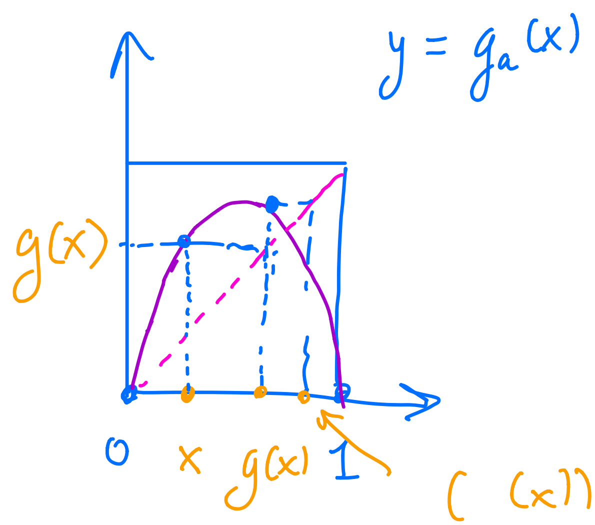
$x \in [0,1]$

$x \in [0, 1]$ "initial condition"

$g_a(x)$ "state of system after 1 unit of time"

$g_a(g_a(x))$ . . . 2 units of time

$g_a(g_a(g_a(x)))$ . . . 3 units. . .



Def: A (DISCRETE-TIME) DYNAMICAL SYSTEM is a pair (X, T), where X is a ... space and $T: X \longrightarrow X$ is a map from X to itself.

Def: $(x, T(x), T(T(x)), \ldots, \overbrace{T \circ \cdots \circ T}^{n}(x), \ldots)$ is the ORBIT of x under g

$T^n(x) := \underbrace{T \circ T \circ \cdots \circ T}_{n}(x)$ n^{th} iterate

E.g.: 1 TOPOLOGICAL DYNAMICS

X = topological space (compact?)

$T: X \circlearrowleft$ continuous.

② MEASURABLE DYNAMICS

$(X, \mathcal{A}, \mu)$ = measure space

$T : X \circlearrowleft$ measure-preserving

$(T_* \mu = \mu)$

$T_* \mu(A) = \mu(T^{-1} A)$

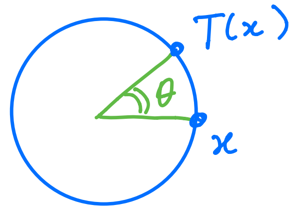
③ HOLOMORPHIC DYNAMICS

X = complex manifold ($\mathbb{C}$)

$T : X \circlearrowleft$ holomorphic map.

Basic Examples

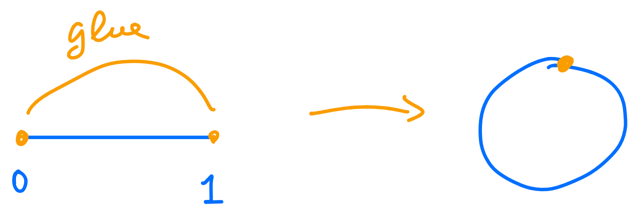
① Rotations



$X = S^1 \simeq \mathbb{R}/\mathbb{Z}$

$T =$ Rotation of angle θ

$(\theta = 2\pi\alpha)$

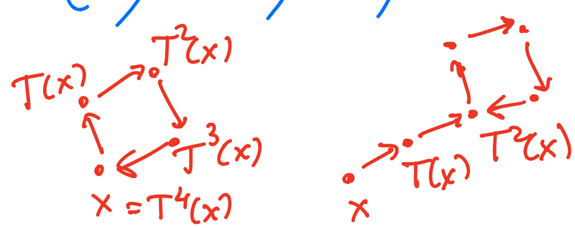


$X = \mathbb{R}/\mathbb{Z}$

$T(x) = x + \alpha \pmod 1$

Denote $\mathcal{O}_T(x) = (x, T(x), \dots, T^n(x) \dots)$

$\alpha = \dfrac{1}{5}$



$$\mathcal{O}_T(x) = \Big(x,\ x + \frac{1}{5},\ x + \frac{2}{5},\ x + \frac{3}{5},\ x + \frac{4}{5},\ \underset{x}{\underset{\shortparallel}{x+1}}\Big)$$

Def.: x is <u>periodic</u> for T if $\mathcal{O}_T(x)$ is <u>finite</u>. It is <u>purely periodic</u> if $\exists\, n \geq 1$ s.t. $T^n(x) = x$ and <u>eventually periodic</u> otherwise

<u>Note</u> : the behaviour of $O_T(x)$ is the <u>same</u> for all x.

<u>E.g.</u>: $\alpha = \frac{p}{q}$, $(p,q)=1$

Then every x is periodic of period q.

<u>What if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$?</u>

No x is periodic.

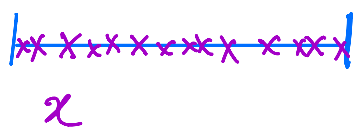
<u>Proof</u> if $T^n(x) = T^m(x)$ for $n > m$

$$x + n\alpha = x + m\alpha \pmod 1$$

$$(n-m)\alpha = 0 \pmod 1$$

$$\Rightarrow \alpha \in \mathbb{Q} \Rightarrow \text{contradiction}$$

Prop.: if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then for any x in $[0,1)$, $\mathcal{O}_{T_\alpha}(x)$ is DENSE in $[0,1)$.



FACT The orbit of x is EQUIDISTRIBUTED along $[0,1)$.

Rotation of irrational angle is a QUASIPERIODIC system.

Example 2

Doubling Map

$$T(x) = 2x \pmod 1$$

$$T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

PERIODIC POINTS for T

$$\frac{1}{2^n} \to \frac{1}{2^{n-1}} \to \cdots \to \frac{1}{2} \to 0 \circlearrowleft$$

$$\frac{1}{3} \to \frac{2}{3} \to \frac{4}{3} = \frac{1}{3}$$

purely periodic period 2

$$\frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5}$$

period 4

$$2^4 - 1 = 15 = 5 \cdot 3$$

period 3

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7}$$

$$7 = 2^3 - 1$$

Fact: Periodic Points for the doubling map are precisely the rationals.

Pf.: $T^n(x) = T^m(x) \quad n > m$

$$2^n x + p = 2^m x + q$$

$(p, q \in \mathbb{Z})$

$$(2^n - 2^m)\, x = q - p$$

$$x = \frac{q-p}{2^m(2^{n-m} - 1)}$$

x purely periodic $(\underline{m = 0})$

$$x = \frac{q-p}{2^n - 1}$$

$\boxed{Q}$ is $\frac{3}{14}$ purely periodic or eventually periodic?

$$\frac{3}{14} \to \frac{3}{7} \to \frac{6}{7} \to \frac{5}{7}$$

$x \in \mathbb{Q} \implies x$ periodic

$x = \frac{p}{q} \implies T^n(x) = \frac{2^n p}{q} + k =$

$$= \frac{2^n p + kq}{q}$$

The orbit of x is contained in the finite set $\left\{ \frac{k}{q},\ k \in \mathbb{Z} \right\}$.

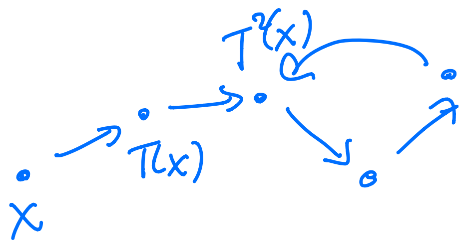
Fact A rational $x \in \mathbb{Q}$ is purely periodic for doubling map iff its denominator is ODD.

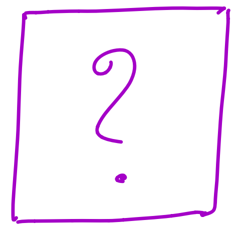
Proof x purely periodic

Then $x = \frac{k}{2^n - 1} \implies \mathrm{den}(x)$ is odd since it divides $2^n - 1$

$x = \frac{p}{q}$, q odd

$\Longrightarrow$ x purely periodic.





<u>Cor.</u>: (1) periodic points are <u>dense</u>

(2) there are points with dense orbit

$\boxed{Q}$ Example of $x \in [0,1)$ with dense orbit for $T(x) = 2x$ mod 1?

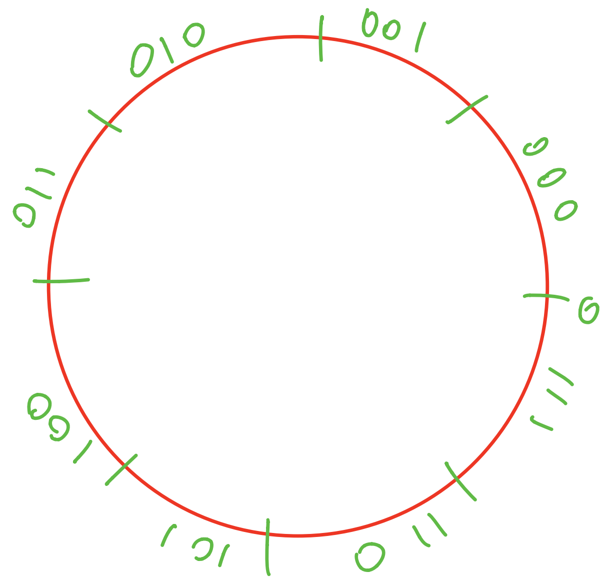
$$x = .b_1 b_2 b_3 \dots b_n \dots \quad \text{(base 2)}$$

$$T(x) = .b_2 b_3 \dots b_n \dots$$

T acts as <u>shift</u>.

$x = .0\ 1\ 00\ 01\ 10\ 11\ 000\ 001 \ldots$

Given any finite sequence, there is n s.t. $T^n(x)$ starts with that sequence. $\Rightarrow$ orbit is DENSE



Def.: T is TOPOLOGICALLY TRANSITIVE if there exists x with dense orbit. T is MINIMAL if EVERY point has dense orbit.

DOUBLING MAP ; ① TOP. TRANSITIVE
(not MINIMAL)

② periodic pts are dense

It is an example of a

HYPERBOLIC/EXPANDING/CHAOTIC

IRRATIONAL ROTATION ① MINIMAL

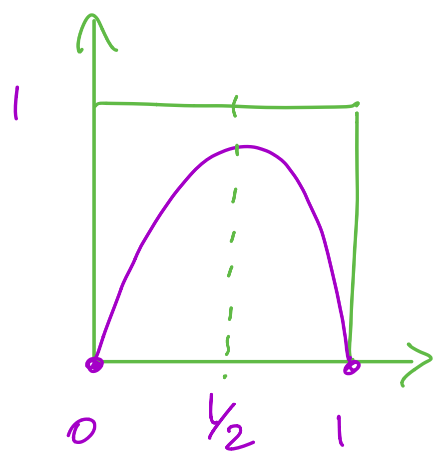
QUASIPERIODIC

Dynamics in One Real Variable

$$g_a(x) = a\,x\,(1-x) = ax - ax^2$$

$a \in \mathbb{R}$ parameter

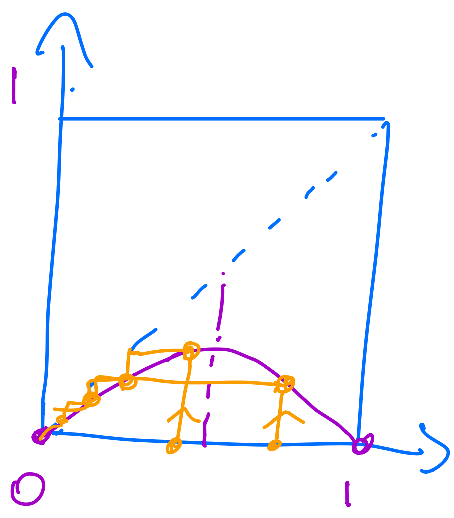
$$g_a : [0,1] \longrightarrow [0,1]$$



$$g_a\left(\tfrac{1}{2}\right) \leq 1$$

$$a\,\frac{1}{2}\cdot\frac{1}{2} \leq 1$$

$$\boxed{0 \leq a \leq 4}$$



a small:

$g_a^n(x) \to 0$

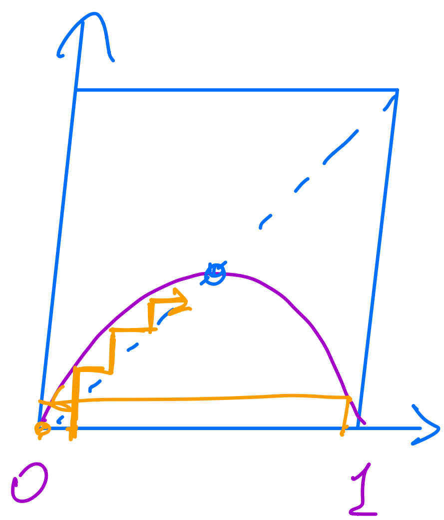
for any x

$g_a'(0) \leq 1$

$g_a'(x) = a - 2xa$

$g_a'(0) = \underline{a \leq 1}$

$\underline{a = 2}$



$g_2\left(\frac{1}{2}\right) = \frac{2}{4}$

$g_2\left(\frac{1}{2}\right) = \frac{1}{2}$

$1 < a \leq 2$

For every x, $g_a^n(x) \longrightarrow x_0 \neq 0$ which is an attracting fixed point.

Def.: x is FIXED POINT if
$$T(x) = x.$$
x is an ATTRACTING F.P. if
$$|T'(x)| < 1$$
x is a REPELLING F.P. if
$$|T'(x)| > 1.$$
In the logistic family $g_a(x)$, for $a \leq 1$, $g_a^n \to 0$

$1+\varepsilon > a > 1$, $g_a^n \to x_0 \neq 0$

Def.: $\underline{a = 1}$ is a <u>BIFURCATION PARAMETER</u>

$$\begin{cases} g_a(x) = ax - ax^2 = x \\ g_a'(x) = a - 2ax \end{cases}$$

$x \neq 0 \qquad a - ax = 1$

$$x = \frac{a-1}{a}$$

$$g_a'(x) = a - 2\cancel{a}\,\frac{a-1}{\cancel{a}} =$$

$$= a - 2a + 2 = |2-a| < 1$$

$$|2-a| < 1 \implies \boxed{1 < a < 3}$$

For $1<a<3$, there is a non-zero attracting fixed point.

GOAL OF CLASS

1. Classification of all local dynamical behaviour for $f_c(z) = z^2 + c$

2. Describing combinatorially Julia & Mandelbrot set

3. Applications to Entropy.
   - using entropies to classify complex polynomials